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Gauge Equivariant Mesh CNNs

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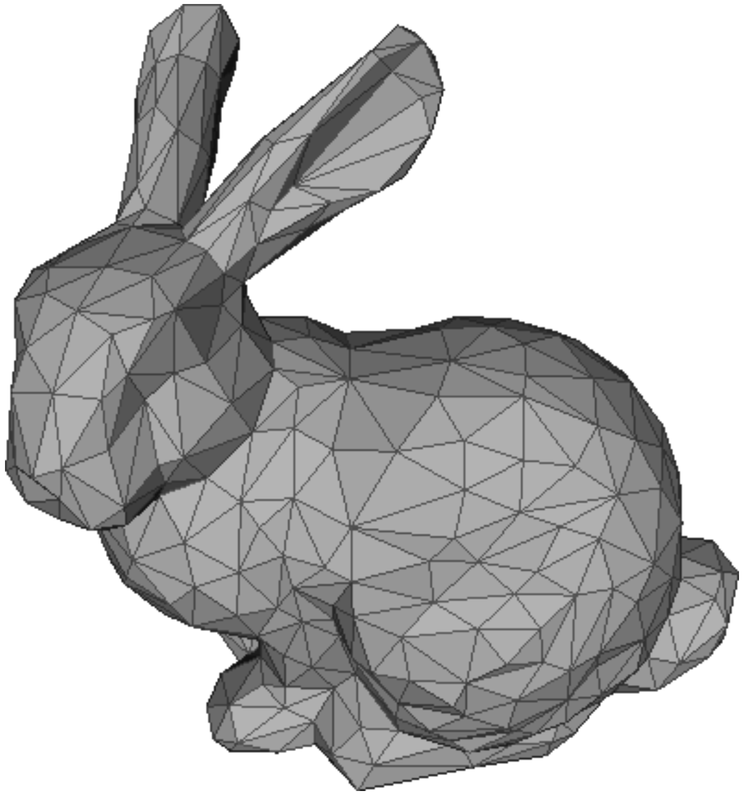
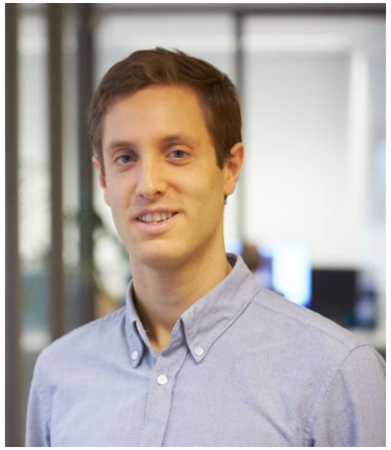
CNN that works on planes and on rabbits

- Orient convolutional kernel
- Gauge – basis of tangent plane
- Gauge Equivariance



Collaboration

Maurice Weiler



Taco Cohen

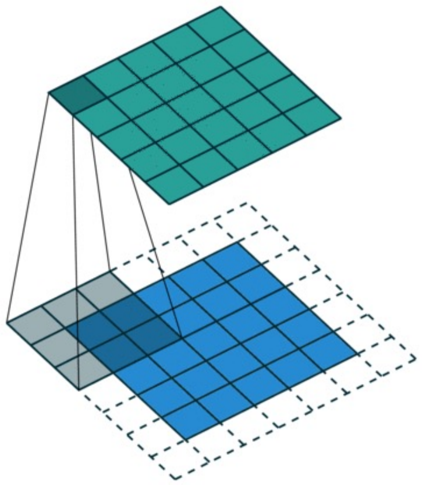
Max Welling

Outline

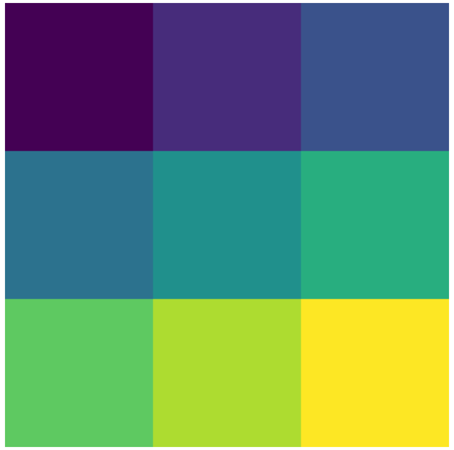
- CNN review
- Message passing on a mesh
- Scalar convolution
- Vector fields
- Gauge equivariant mesh convolutions
- Implementation
- Application to blood flow

Convolutional neural networks on images

- Image feature $f \in \mathbb{R}^{L \times L}$
- Kernel $k \in \mathbb{R}^{3 \times 3}$
- $f_p' = (k \star f)_p = \sum_q k(q) f(p - q)$
- Alternate convolutions with non-linearities
- Learn kernels k

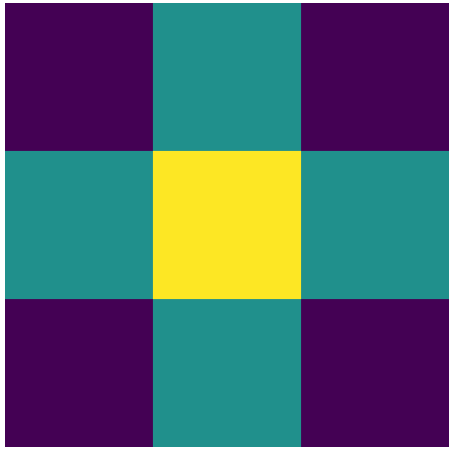


Anisotropy



Anisotropic

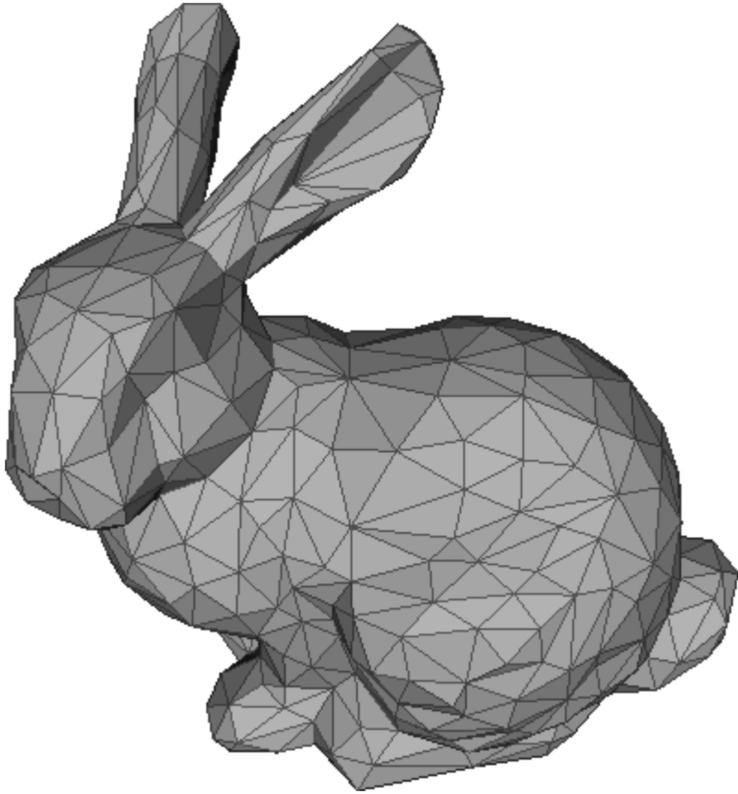
- 9 parameters
- Detects any edge



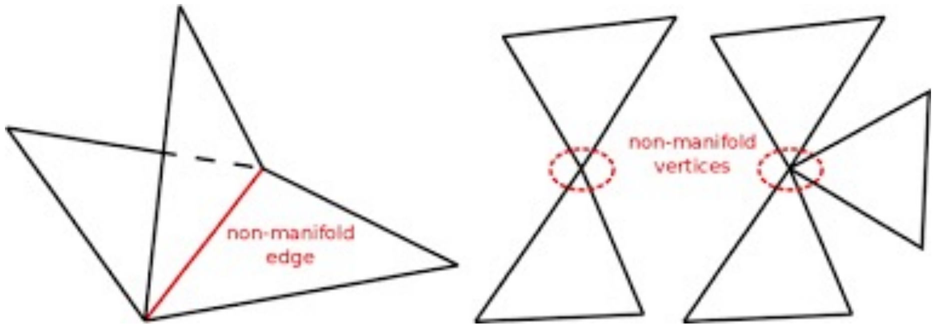
Isotropic

- 3 parameters
- Can not detect edges

Mesh



- Discretization of curved surface
- Triangular mesh = collection of triangular faces
- Manifold mesh: connected faces look like a plane

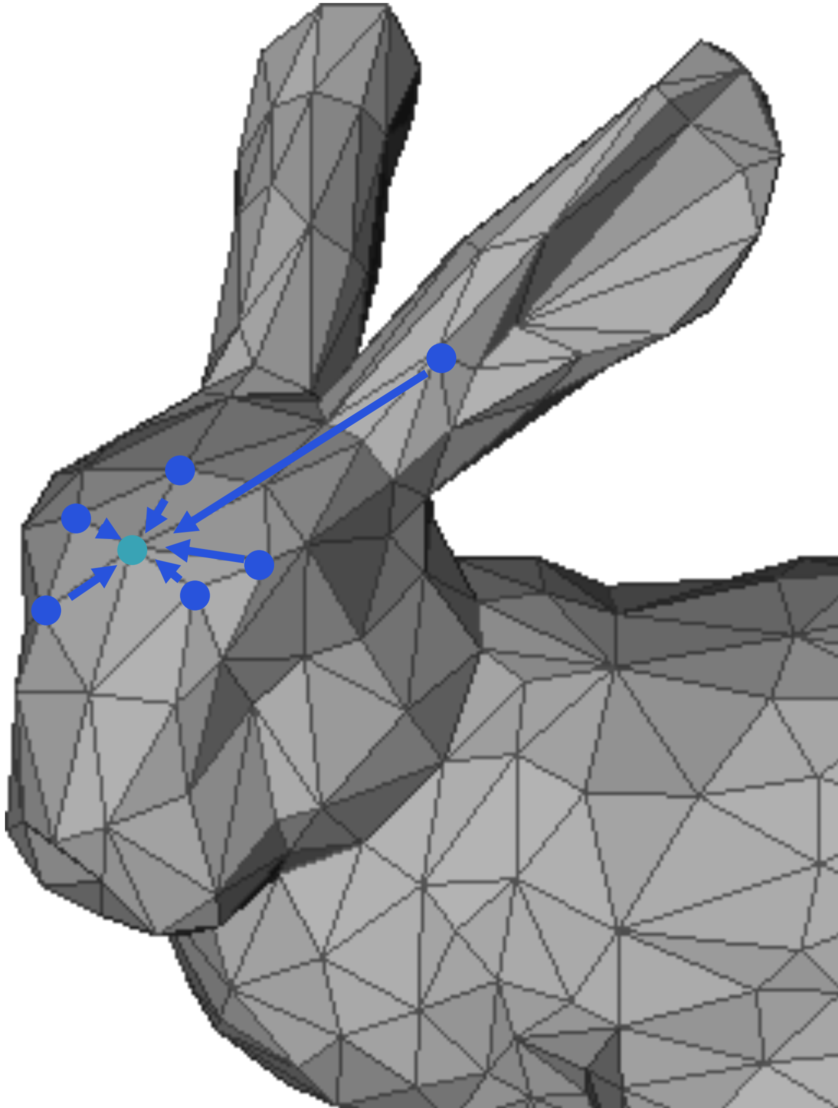


Message passing on a mesh

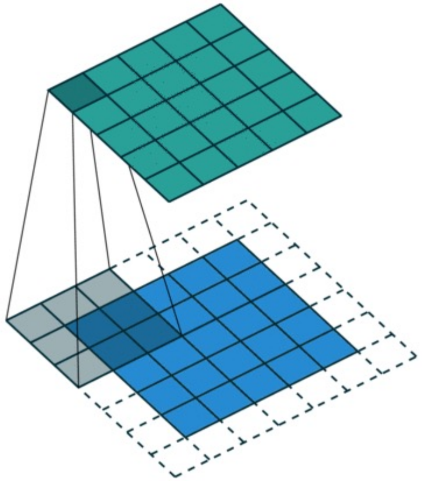
- Feature at vertices
- Message passing

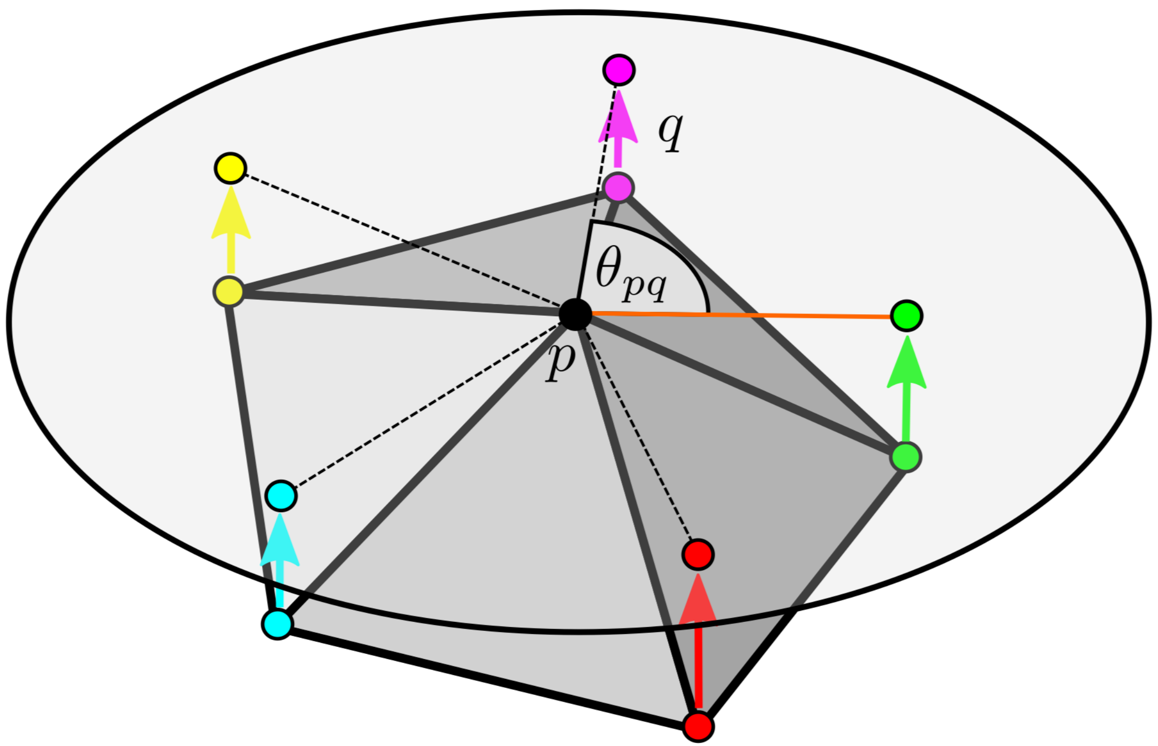
$$f_p' = \sum_{q \in \mathcal{N}(p)} k(q \to p) f_q$$

- Applications:
  - Segment vertices
  - Classify shapes
  - Predict blood flow
- Initial features
  - Vertex coordinates
  - Local description of curvature



Convolutions on a mesh



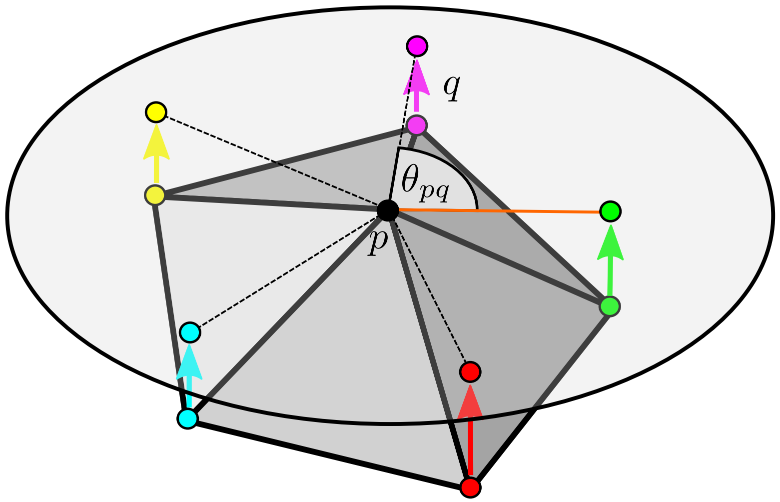


- Canonical relative (x, y) coordinates of neighbours

- Log map to tangent plane
- Polar coordinates
- What is $\theta = 0$?
- Choice of coordinates: gauge

Gauge invariance, fixing & equivariance

- Gauge: choice of basis for each tangent plane
  - Reference neighbour
- Option 1: gauge invariance
  - Message $q \to p$ independent of θ_{pq}
  - But: isotropic
- Option 2: gauge fixing
  - Principal curvature direction
  - But: ill-defined
- Option 3: Gauge equivariance [Cohen et al. 2019]:
  - The same feature in different gauges has same output (up to rotation)

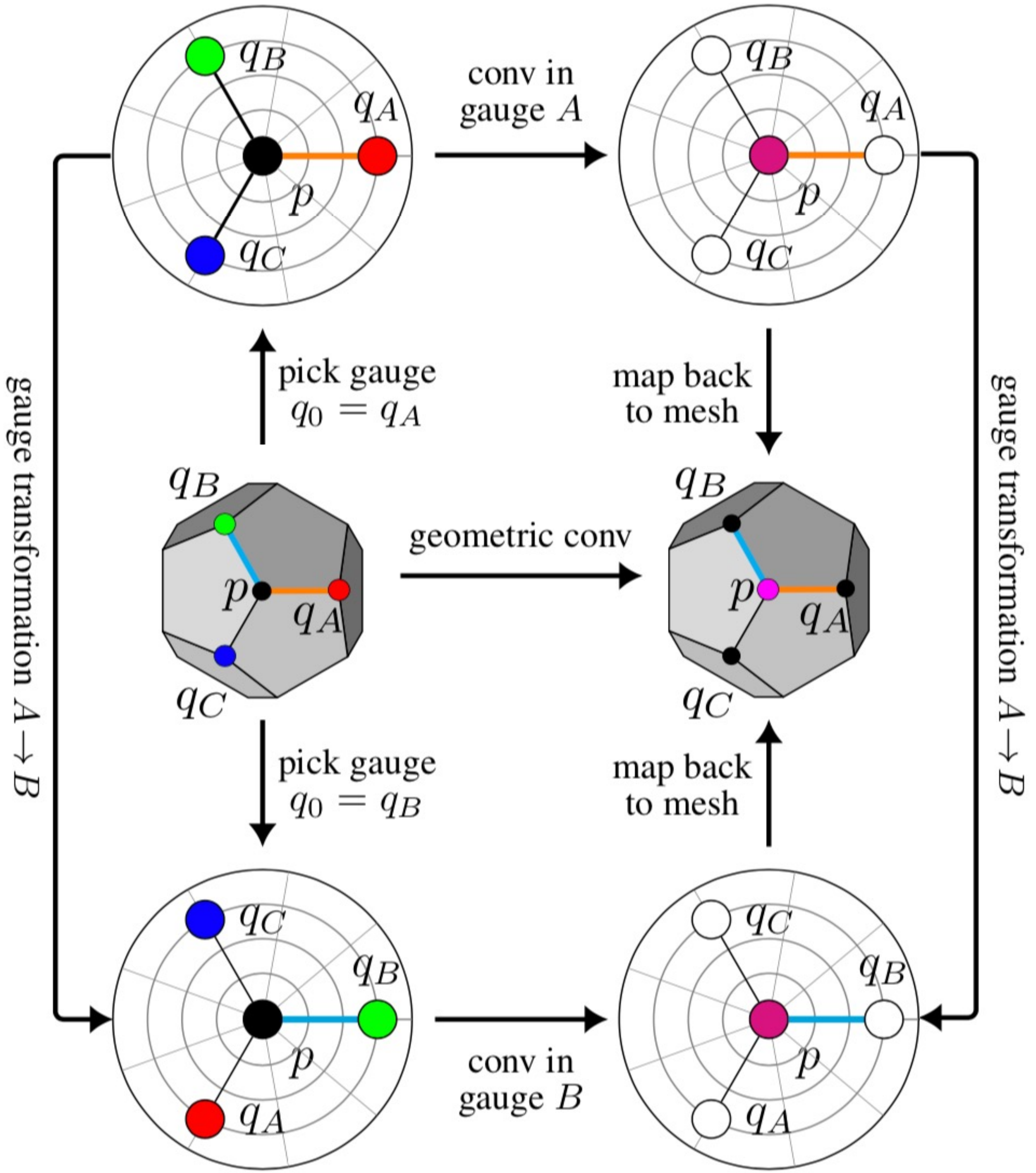


Gauge equivariance with scalar features

- Feature $f: M \to \mathbb{R}$
- Transformation rule under gauge transformation: invariant
- Gauge w, polar coordinates of neighbour q of p: $w_p(q) = (r_q, \theta_q)$
- Different gauge w', has coordinates $w'_p(q) = (r_q, \theta'_q) = (r_q, \theta_q + g_p)$
- Kernel $K(r, \theta) \in \mathbb{R}$
- In gauge w: $(K \star f)_p = \sum_{q \in \mathcal{N}(p)} K(r_q, \theta_q) f_q$
- In gauge w': $(K \star f)_p = \sum_{q \in \mathcal{N}(p)} K(r_q, \theta_q + g_p) f_q$
- Equality for any angle g_p implies $K(r_q, \theta_q + g_p) = K(r_q, \theta_q)$
- Kernel isotropic

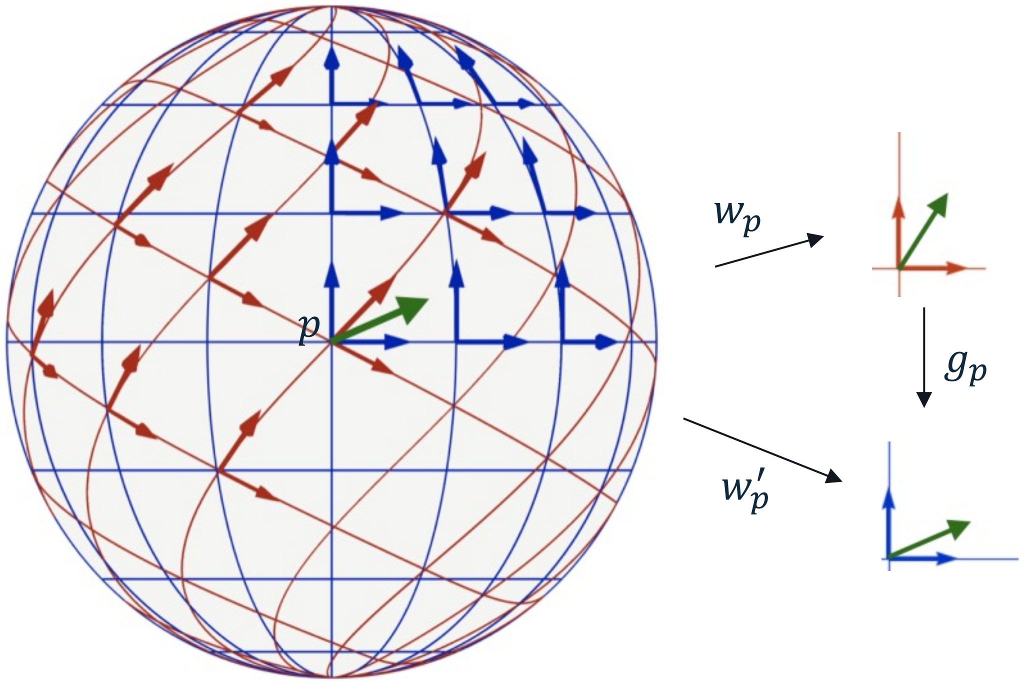
Gauge equivariant convolutions on scalar fields

Scalar convolutions are isotropic



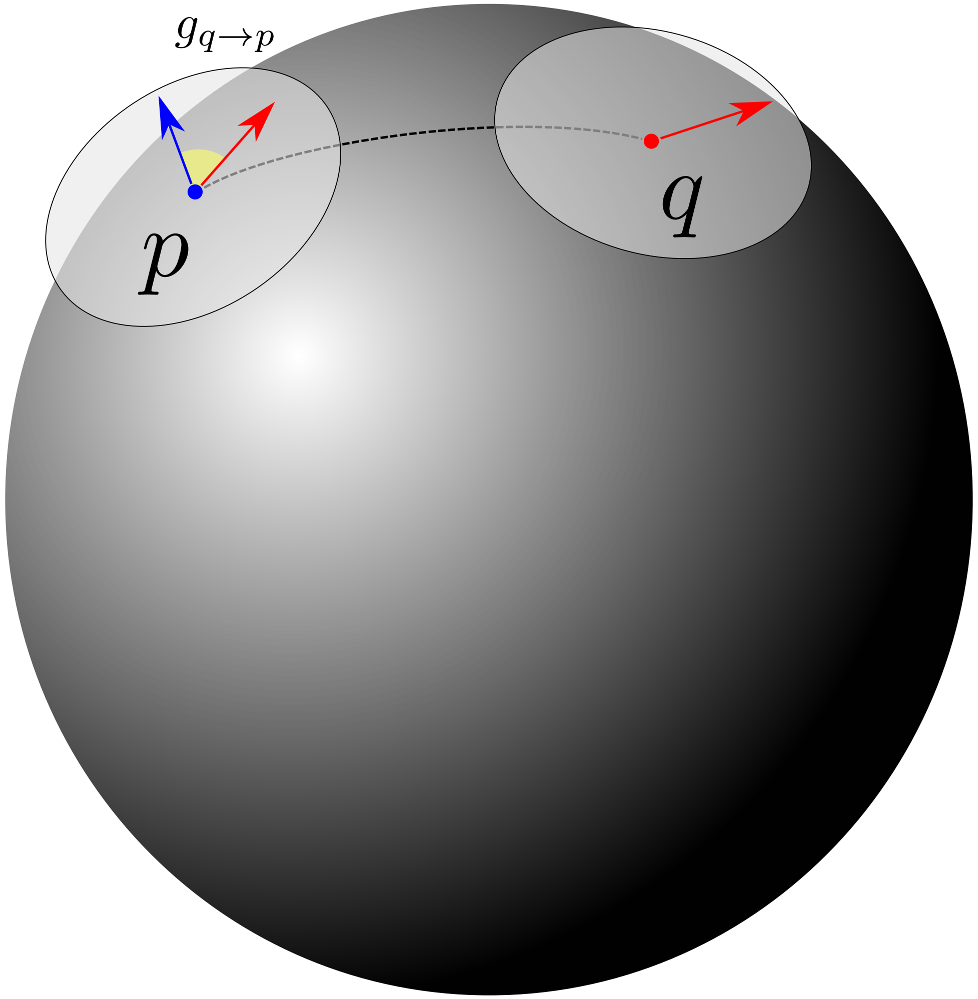
Tangent vector field features

- Feature $f_p \in T_pM$
- In gauge w_p, $f_p \in \mathbb{R}^2$
- Let $w_p'(q) = w_p(q) + g_p$
- In gauge w_p', $f_p' = \rho(-g_p) f_p$
- $\rho(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
- More general feature: ρ group representation of group of planar rotations $SO(2)$



Parallel Transport

- Tangent planes not parallel
- Parallel transport of geodesic
- Transport gauge-defining X-axis
- Angle $g_{q \to p}$
- Cheaply precomputed
- Any parallel transport by linearity

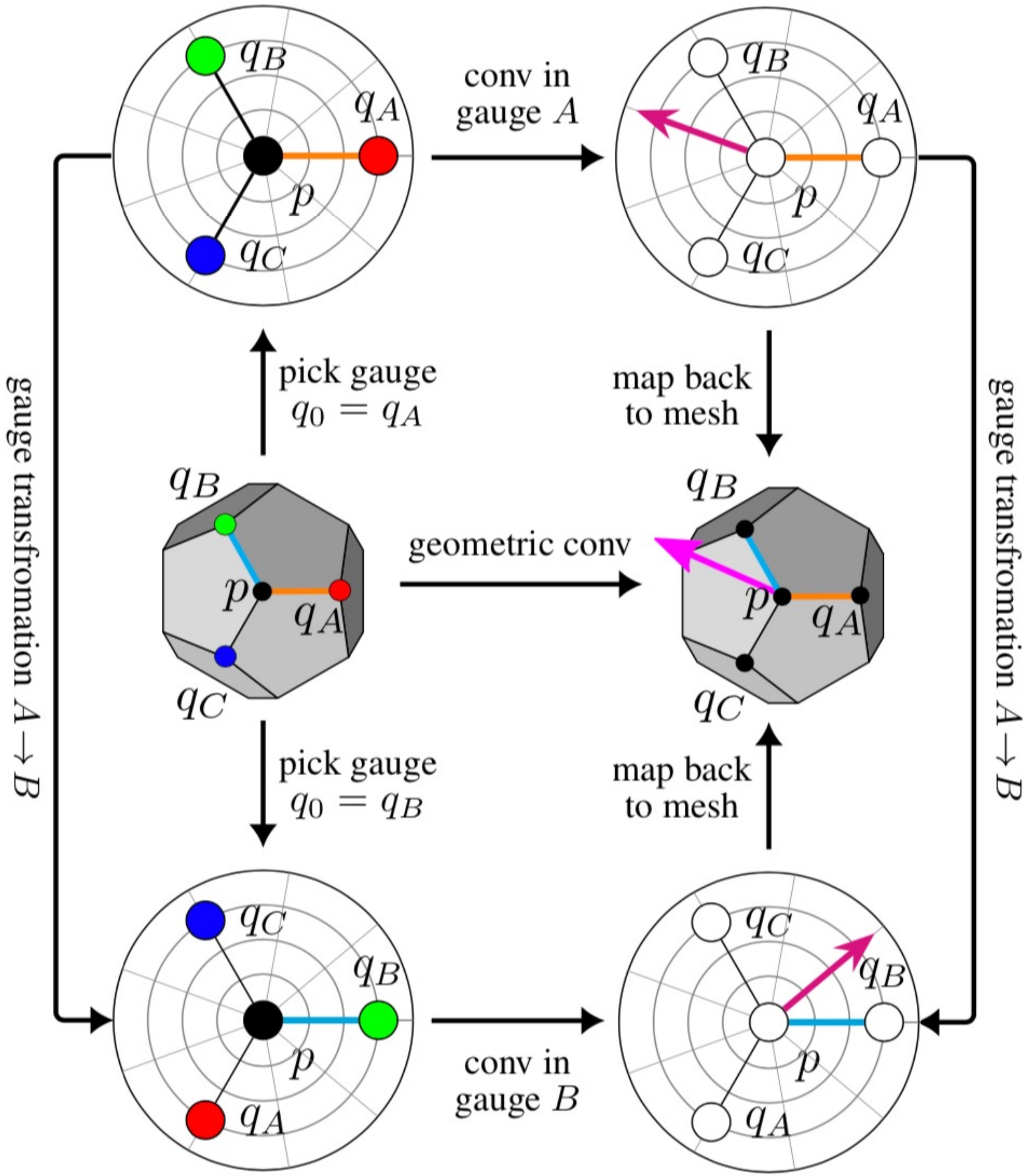


General Gauge Equivariant Convolution

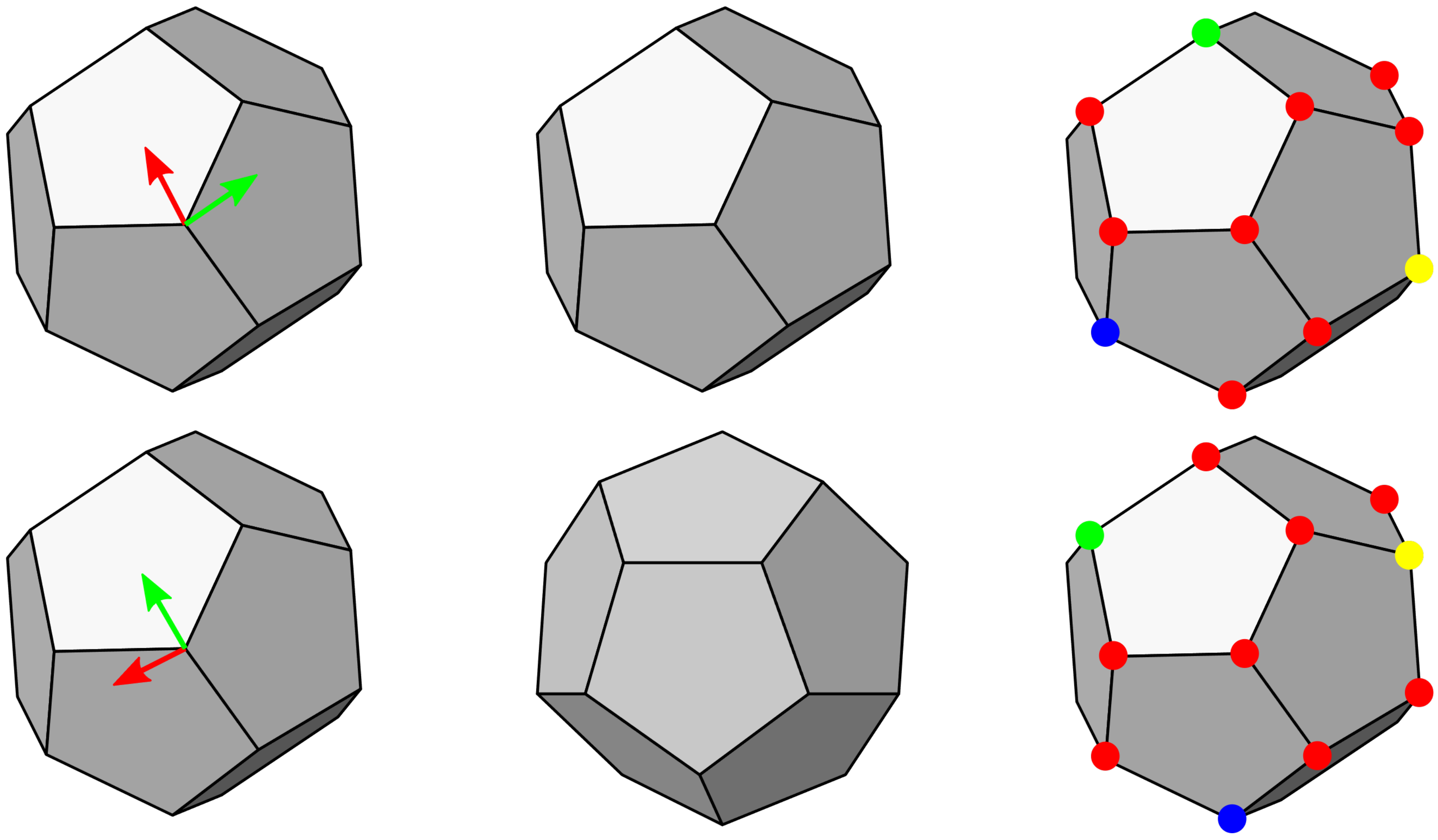
- Two gauges are related by planar rotation $g \in SO(2)$
- Vertex feature: group representation $\rho(g) \in \mathbb{R}^{d \times d}$
  - E.g. scalar feature $\rho(g) = 1$
  - E.g. tangent vector feature $\rho(g) = \begin{pmatrix} \cos(g) & -\sin(g) \\ \sin(g) & \cos(g) \end{pmatrix}$
- Kernel $K(r, \theta) \in \mathbb{R}^{d' \times d}$
- Convolution: $(K \star f)_p = \sum_{q \in \mathcal{N}(p)} K\left(r_q, \theta_q\right)\rho(g_{q \to p}) f_q$
- Equivariance if: $\rho'(g)K(r, \theta) = K(r, \theta + g)\rho(g)$

Gauge equivariant convolutions on vector fields

Vector convolutions are anisotropic



Symmetry properties



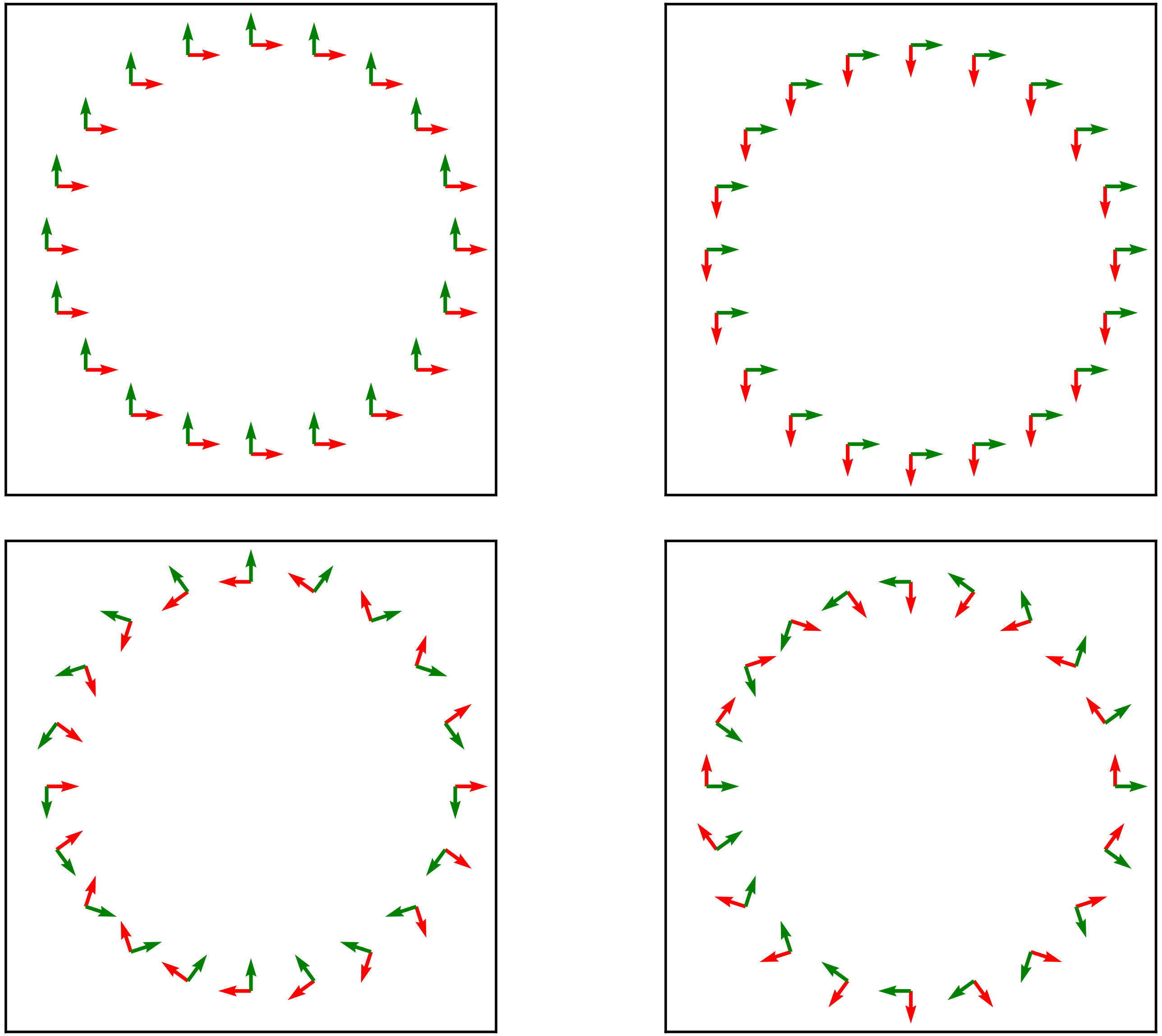
Gauge

Vertex coordinates

Mesh isometries

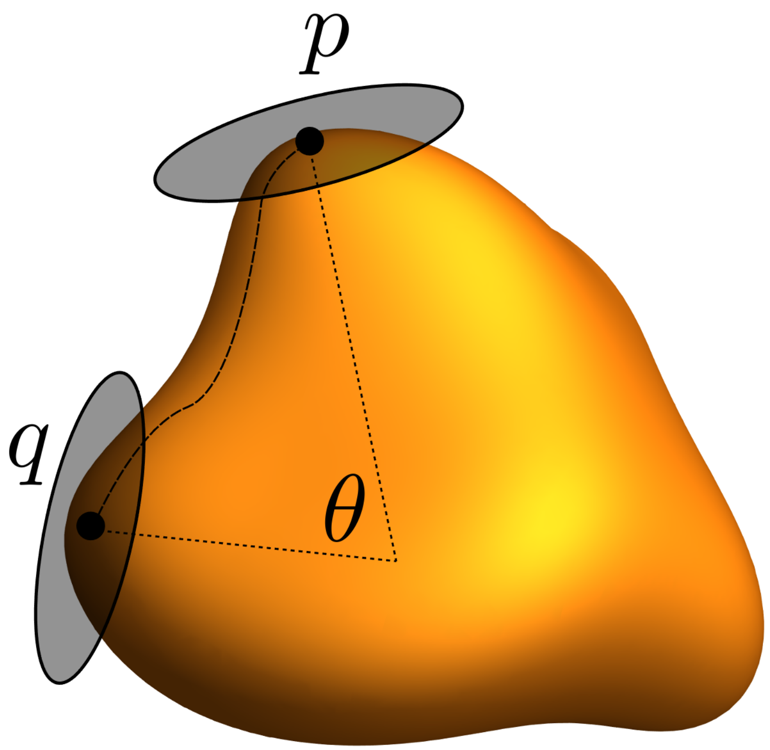
Solving the kernel constraint

- Constraint
  $$\rho'(g)K(r,\theta) = K(r,\theta+g)\rho(g)$$
- For $r > 0$
  - Freely choose $K(r_1,0),\ldots,K(r_N,0)$
  - Linearly interpolate for $K(r,0)$
  - $K(r,\theta) = \rho'(\theta)K(r,0)\rho(-\theta)$
- For $r = 0$
  - Linear constraint
    $$\rho'(g)K(0) = K(0)\rho(g)$$
  - Linearly combine solution with learned parameters



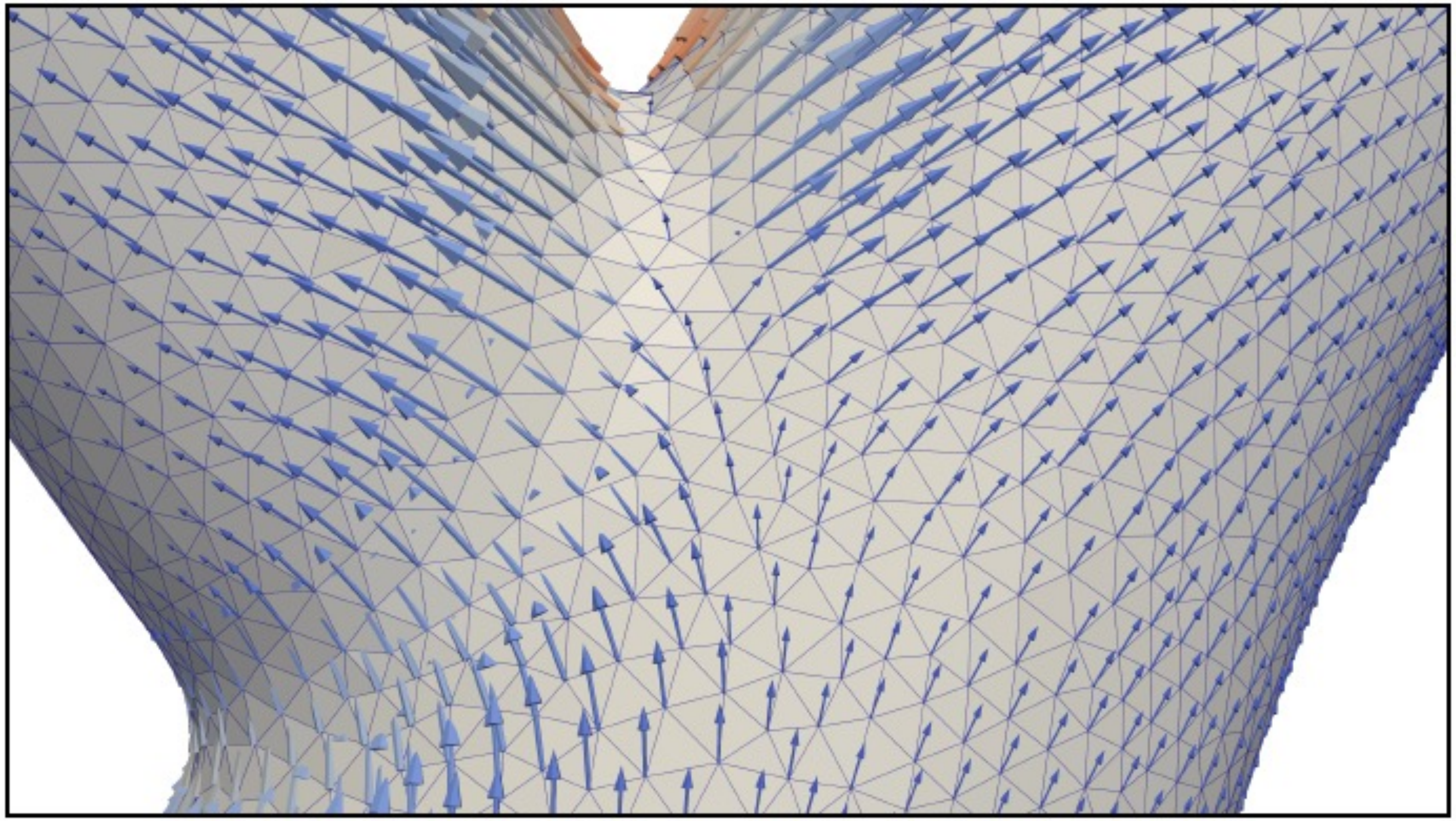
Efficiently approximating geometry

- Geometric quantities
  - Logarithmic map
  - Parallel transport
- Exact: Partial Differential Equation
- Spherical approximation



Application to blood flow

[Suk, de Haan, Lippe, Brune, Wolterink, 2021]



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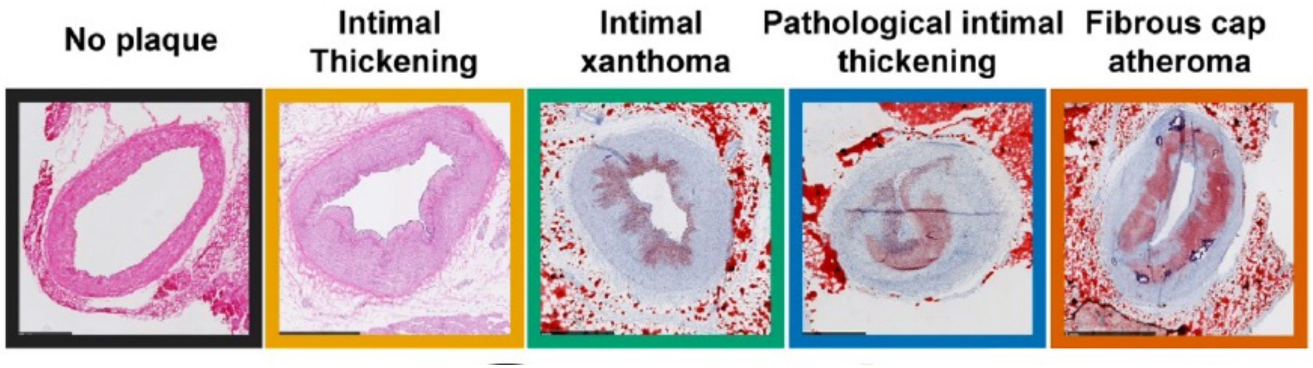
University of Twente

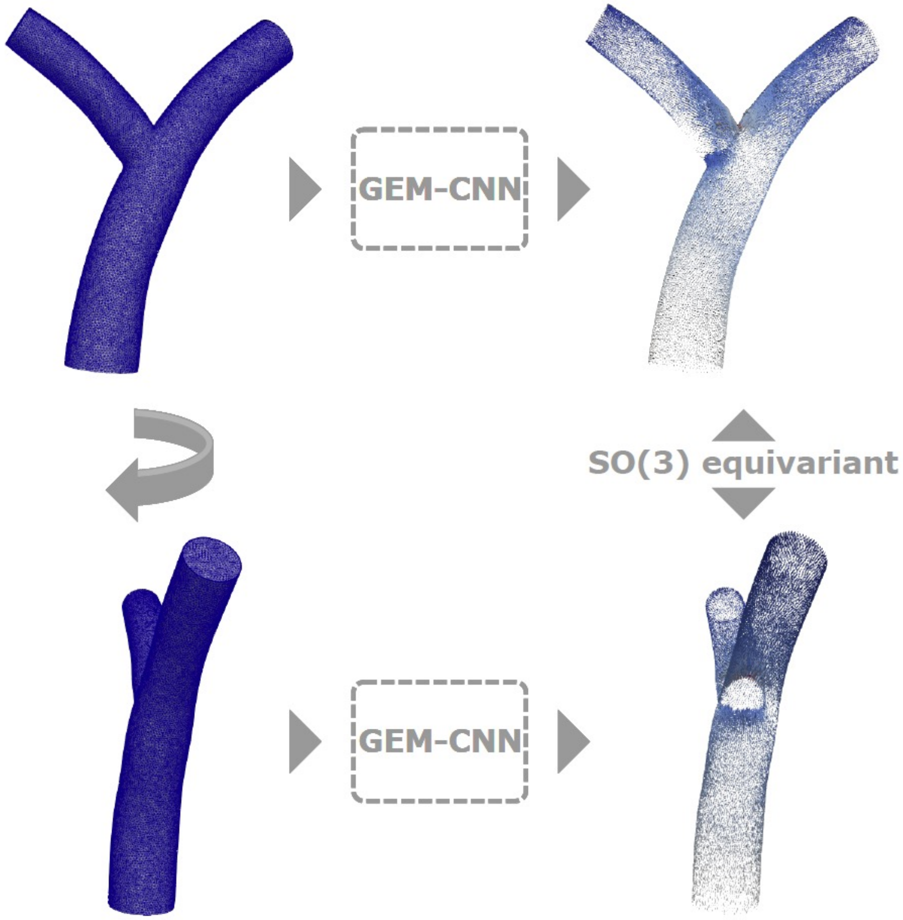
Problem formulation

- Shape of arteries in human body related to e.g. aneurysm
- Quantitative analysis of blood flow useful indicator – wall shear stress
- Non-invasively: model artery with MRI scanner
- Simulate blood flow with computational fluid dynamics (> 20h)
- Learn neural network surrogate to predict WSS on CFD ground truth
- Dataset of realistic random meshes

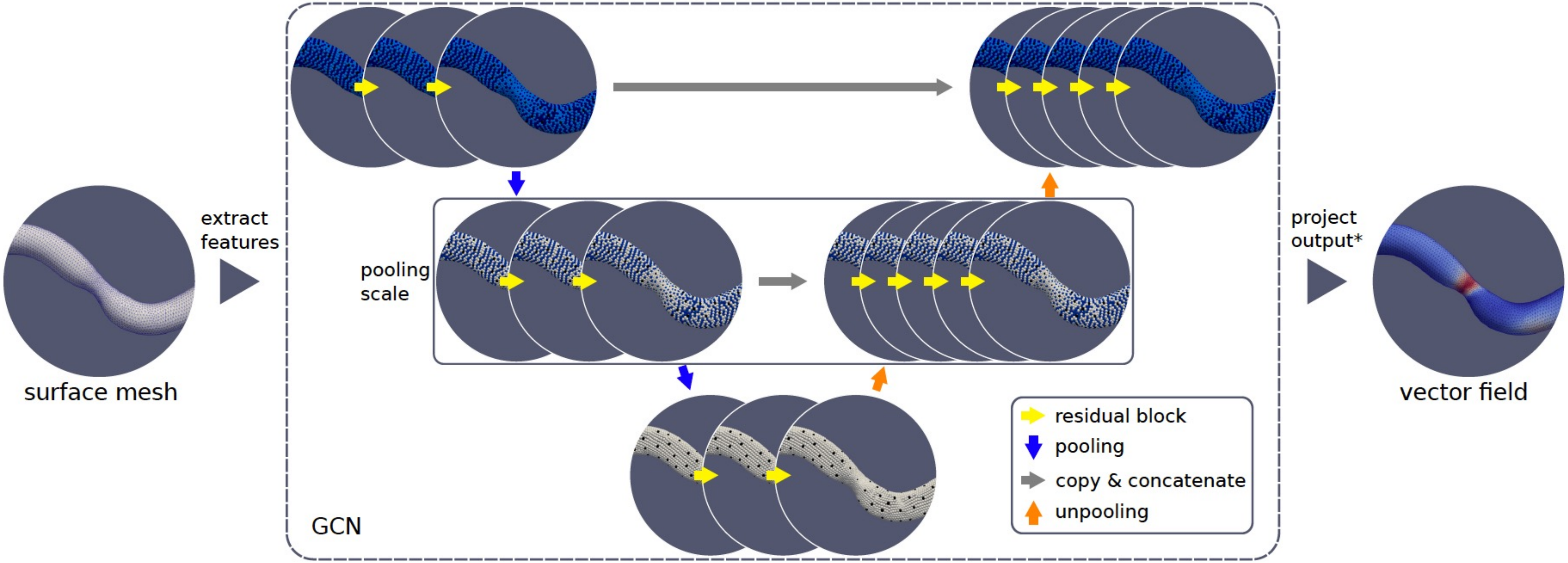


Equivariance

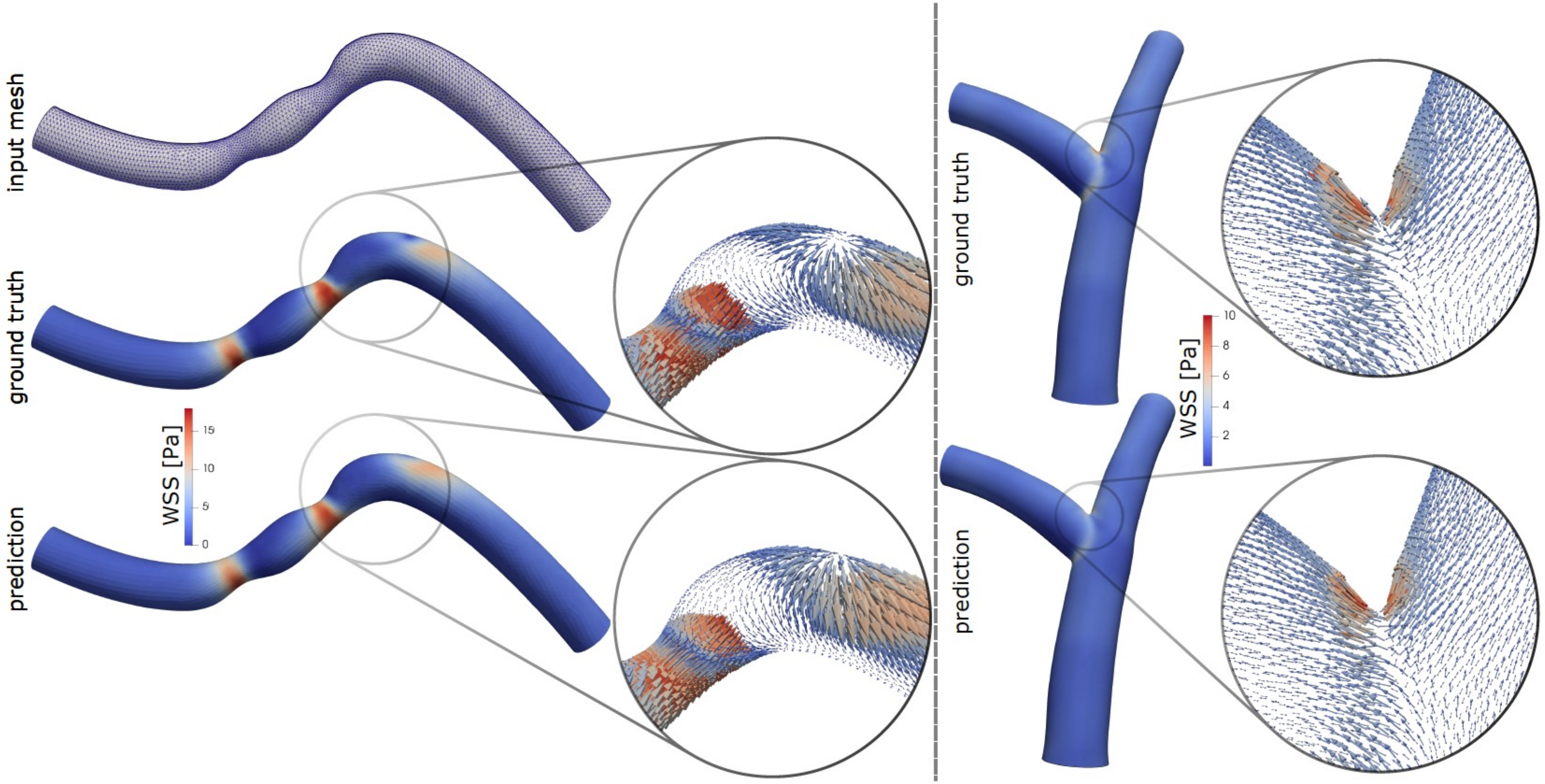
- Arteries not in canonical orientation
- Equivariance to global transformations
- Gauge Equivariant Mesh CNN



Network Architecture



Results



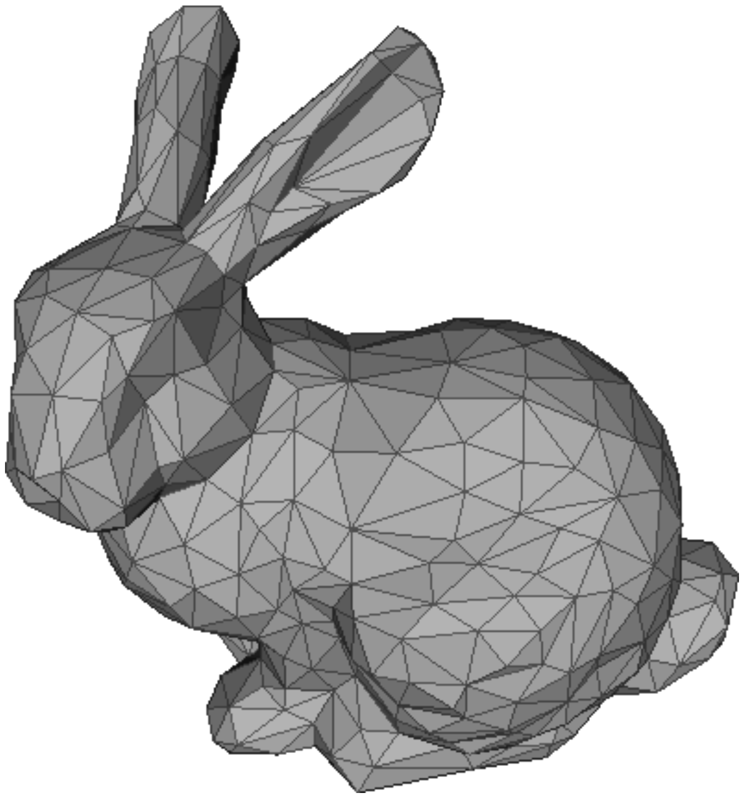
		NMAE [%]	$\triangle_{max}$ [Pa]
Single Arteries	SAGE-CNN	2.0	7.80
	FeaSt-CNN	1.1	5.13
	GEM-CNN	**0.6**	**3.68**
	SAGE-CNN†	9.6	23.96
	FeaSt-CNN†	7.5	22.93
	GEM-CNN†	0.6	3.39
Bifurcating Arteries	SAGE-CNN	1.1	4.14
	FeaSt-CNN	**0.9**	3.72
	GEM-CNN	1.1	**3.71**
	SAGE-CNN†	6.9	8.29
	FeaSt-CNN†	7.4	8.33
	GEM-CNN†	1.1	3.67

† evaluated on randomly rotated data

Takeaway

Gauge Equivariant Mesh CNN is:

- Simple
- Scalable
- Anisotropic ⇒ expressive
- Symmetry properties
- Try it out:
  github.com/Qualcomm-AI-research/gauge-equivariant-mesh-cnn



Thank you

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